Exponential and Logarithmic Functions

Finite Math

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Definition

Definition (Exponential Function)

An exponential function is a function of the form

$$f(x) = b^x, \ b > 0, \ b \neq 1.$$

b is called the base.

Definition

Why the restrictions on b?

- If $b = 1$, then $f(x) = 1^x = 1$ for all x values. Not a very interesting function!
- As an example of the case when $b < 0$, suppose $b = -1$. Then

$$f\left(\frac{1}{2}\right) = (-1)^{1/2} = \sqrt{-1} = i$$

  an imaginary number! This kind of thing will always happen if b is negative.
- If $b = 0$, then for negative x values, f is not defined. For example,

$$f(-1) = 0^{-1} = \frac{1}{0} = \text{undefined.}$$

Graphing Exponential Functions

Example

Sketch the graph of $f(x) = 2^x$.

Graphing Exponential Functions

When $b > 1$, the graph of $f(x) = b^x$ has the same basic shape as 2^x, but may be steeper or more gradual. Let's see what happens when $b < 1$.

Example

Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

Negative Powers

Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

Properties of Exponential Functions

Property (Graphical Properties of Exponential Functions)

The graph of $f(x) = b^x$, $b > 0$, $b \neq 1$ *satisfies the following properties:*

1. *All graphs pass through the point* $(0, 1)$.
2. *All graphs are continuous.*
3. *The* x*-axis is a horizontal asymptote.*
4. b^x *is increasing if* $b > 1$.
5. b^x *is decreasing if* $0 < b < 1$.

Properties of Exponential Functions

Property (General Properties of Exponents)

Let $a, b > 0$, $a, b \neq 1$, and x, y be real numbers. The following properties are satisfied:

1. $a^x a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
2. $a^x = a^y$ *if and only if* $x = y$
3. $a^x = b^x$ *for all* x *if and only if* $a = b$

The Natural Number

There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which the quantity

$$\left(1 + \frac{1}{x}\right)^x$$

approaches as x tends towards ∞.
This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Growth and Decay Example

Example

In 2013, the estimated world population was 7.1 *billion people with a relative growth rate of* 1.1%.

(a) *Write a function modeling the world population* t *years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Now You Try It!

Example

The population of some countries has a relative growth rate of 3% *per year. Suppose the population of such a country in 2012 is* 6.6 *million.*

(a) *Write a function modeling the population* t *years after 2012.*

(b) *What is the expected population in 2018? 2022?*

Solution

(a) $P = 6.6e^{0.03t}$

(b) 7.90 *million;* 8.91 *million*